A STOCHASTIC MODEL FOR ARTIFICIAL EARTH SATELLITE ILLUMINATION

ROBERT D. KAISER

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by

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Lieutenant Commander, United States Navy

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ABSTRACT

The illumination condition of an artificial earth satellite is investigated in relation to the geographic location of the sub-satellite point. A stochastic problem is formulated by postulating a "random" satellite, and the solution to this problem is indicated analytically. A statistical solution is obtained from a digital computer simulation program, together with a sensitivity analysis of parametric variations.

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PREFACE

The general purpose of this thesis is to investigate certain aspects of the illumination of artificial earth satellites by the sun, in relation to the track of the sub-satellite point as measured in a geographic coordinate system. Imposed upon this study is a stochastic model of the earth-satellite system.

An analytic approach to the basic problem was made, followed by a computer simulation program for a statistical solution. The basic problem is set forth in the Introduction.

I wish to express my appreciation to the following persons for their contributions to this study:

Mr. E. J. Leinfelder, Chance-Vought Corp., Astronautics Division, who first posed the problem to me, and who provided the stimulus and inspiration leading to this study;

Professor W. P. Cunningham, U. S. Naval Postgraduate School, who indicated the method of formulating the problem in a manner which led to the desired results;

Professor R. M. Thatcher and Miss Pat Hoang, U. S. Naval Postgraduate School, who assisted me in no small measure in the design, writing and de-bugging of the computer simulation program; and

Professor R. R. Read, U. S. Naval Postgraduate School, who detected initial errors in the chapter on probability formulation.

GLOSSARY

Symbol	Definition
a	semi-major axis of orbital ellipse
dl	apparent change in longitude of sub-satellite point from initial reference time to time t
h	altitude of satellite above surface of earth
i	inclination of satellite orbital plane to equatorial plane
i'	inclination of equatorial plane to ecliptic plane
W	longitude of designated geographical points
l'_t	longitude of sub-satellite point at time t
l'_o	longitude of ascending node
$\bar{r}$	unit vector from center of earth to satellite
r'	projection of $\bar{r}$ onto equatorial plane
s	specified angular distance of sub-satellite point from designated geographical points
t	instantaneous time after initial reference time
w_e	angular rotational velocity of earth about polar axis in radians per unit time
w_s	angular velocity of satellite in radians per unit time
A	angle between $\bar{E}$ and $\bar{r}$ vectors
C	angle at satellite subtended by the center of the earth and the surface of the earth
D	angle between vernal equinox and ascending node of satellite

Symbol	Definition
$\overline{E}$	unit vector from center of earth to sun
F	angle between ascending node and projection of $\overline{r}$ onto equatorial plane
G	angle between ascending node and $\overline{r}$ vector
G'	universal gravitational constant
H	angle between vernal equinox and $\overline{E}$ vector
L	latitude of designated geographical points
L'	angle between $\overline{r}$ and r'
M	event that desired passage occurs during first orbit
M'	mass of the earth
N	event that desired illumination occurs at time of closest approach to designated geographical point
R_e	radius of the earth
T	orbital period of satellite

INTRODUCTION

It is proposed to investigate some aspects concerning the illumination of an artificial earth satellite by the sun in relation to the track of the sub-satellite point as measured in a geographic coordinate system. But before stating the complete problem, some preliminary discussion is necessary.

Consider a satellite in a circular orbit around the earth. Its position may be completely specified at any time (ignoring perturbations) if its altitude, orbital inclination to the equatorial plane, and the longitude of its ascending node at a particular time is known. The longitude of the ascending node may be defined as the longitude at which the sub-satellite point crosses the equator entering the northern hemisphere. Under these circumstances the track of the sub-satellite point on the earth's surface may be computed for any set of sequential orbits.

Due to the rotation of the earth about its polar axis, the longitude of the ascending node is displaced westward on each orbit by an amount equal to the rotational velocity of the earth times the orbital period of the satellite. This displacement will be denoted $w_e T$, where w_e is the angular rotational velocity of the earth about its polar axis, and T is the orbital period of the satellite.

If it is desired to determine if the track of the sub-satellite point will pass within a given distance of a given point on the earth's

surface, then under the stated conditions the problem is completely deterministic. A simple but somewhat tedious calculation will ascertain the number of orbits required, the exact time, and the actual distance from the given point at the time of closest approach.

Now suppose that, instead of being completely specified, these parameters of altitude, inclination, and longitude of ascending node were random variables. Conceivably this situation could arise where a satellite has been "lost" to surface detection units for an extended time period. Similarly, this proposition might be required in a feasibility study of a ground visual detection station. In any event, under these circumstances the problem of ascertaining if the satellite track passes within a given distance of a given point is no longer deterministic but stochastic. The solution must be given as a probability of such passage after a given number of orbits, and will be a function of these three random variables.

To simplify the problem considerably, let the variables of altitude and inclination take on a range of discrete values. Then seek a solution for the random variable of longitude of ascending node for each pair of values for altitude and inclination. The longitudes of the second and all subsequent ascending nodes are simple functions of the first nodal longitude, since specifying the altitude specifies orbital period, hence specifies the longitude displacement, $w_e T$. Then the probability of passage is a function of the random variable

of longitude of first ascending node, and of the orbital period, T.

The second part of the problem is to determine if the satellite is also illuminated by the sun. It appears obvious that the satellite need not be over the sunlit side of the earth to be illuminated. In fact, the only requirement for illumination is that the satellite be outside the shadow cone of the earth. For satellites at sufficient altitudes this may not occur until the satellite is well into the "back side" of its orbit.

With these considerations in mind, the main problem may now be stated:

Given a satellite at an altitude h, with an orbital inclination i, and two geographical points on the earth's surface on the same meridian of longitude W, and equidistant from the equator at latitudes L North and South:

Find the joint cumulative probability that the satellite will pass within a given angular distance s of either point and that the satellite will simultaneously be illuminated by the sun, the result to be stated as a cumulative probability distribution function in terms of successive orbital periods after initial reference time (arbitrarily set equal to zero).

Some previous studies of satellite illumination have been made, made, but no studies have been found which treat the problem from a stochastic viewpoint.[1,2]

This thesis first develops the necessary mathematical models for the orbital mechanics, then sets forth the precise probability formulation statements. The statistical solutions are then derived from a digital computer simulation program which is discussed in the appendices.

While the design and writing of the computer simulation program proved of great interest to the author, and provided an opportunity to apply recently acquired knowledge of Monte Carlo techniques, it must be emphasized that the computer program is not an entity in itself, but merely a useful tool in arriving at a solution to the stated problem.

[1] Leinfelder, E.J., and Thrasher, J.R., Orbit-Earth Shadow Relations, AST/EOR-13277, Chance-Vought Corp., Dallas, Texas, 1960. CONFIDENTIAL

[2] Koelle, H.H., Editor, Handbook of Astronautical Engineering, McGraw-Hill Book Co., New York, 1961.

CHAPTER I

ORBITAL MECHANICS

Initial Assumptions

1. The earth's solar orbit and the satellite orbit have zero eccentricity.

2. The earth is a perfect sphere with total mass concentrated at the center.

3. The earth's equatorial plane is inclined to the ecliptic plane at a constant angle i' (= 23° 26' 59").

4. The earth's solar umbra is a circular cylinder (valid for values of h less than three times the earth's radius).

5. At initial reference time the longitude of the ascending node of the satellite may be considered a random variable, uniformly distributed on the interval $[0, 2\pi]$. At this time the satellite will be assumed to be located at the ascending node.

6. All other perturbations of a satellite orbit will be ignored. This includes atmospheric drag and solar radiation pressure.

Discussion of Assumptions

The effect of these assumptions is to confine the problem to one involving only central forces. This also implies that precession of the satellite orbital plane will not be included. In order that these omissions do not radically affect the solution, the probability distribution function sought will only be examined for a period of 16 orbits.

This will be on the order of 24 hours for the range of altitudes considered. As will be seen subsequently, two more random variables must be introduced; one to account for the space orientation of the satellite orbital plane, and another to account for seasonal variations.

<u>Method of Attack</u>

A mathematical model will be established representing the motion of the satellite and the sun with reference to a geocentric coordinate system, followed by a probability formulation.

<u>Mathematical Model</u>

Let $\overline{E}$ represent the vector from the center of the earth to the sun, and let $\overline{r}$ represent the vector from the center of the earth to the satellite. The initial assumptions will ensure that these vectors are of constant magnitude and possess constant rotational velocities about the center of the earth.

Using Figure I-1, find the direction cosines of $\overline{E}$ in the rectangular geocentric coordinate system depicted, where the X-axis is along the vernal equinox, the Z axis is directed through the north celestial pole, and the Y-axis perpendicular to the other two axes in a conventional manner. The origin is, of course, the center of the earth.

During a yearly cycle the angle $\underline{H}$ takes on all values in the interval $\left[0, 2\pi\right]$. The values of x, y, and z are easily determined as:

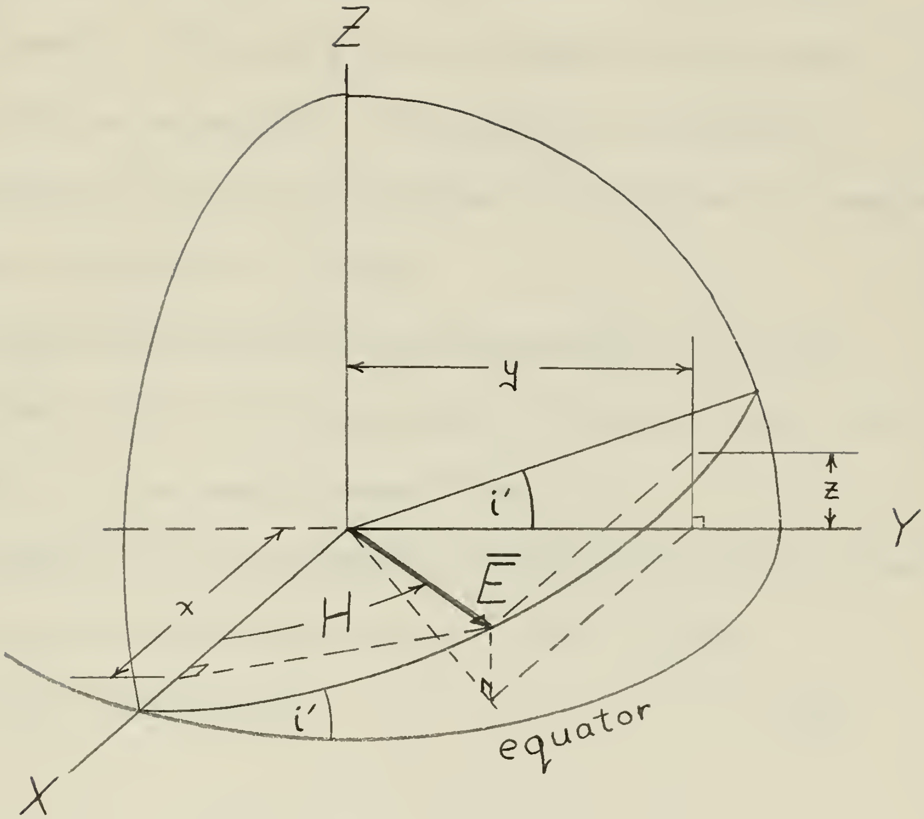

Fig. I-1: Representation of Earth-Sun vector $\overline{E}$

$$\left.\begin{aligned} x &= E \cos H \\ y &= E \cos i' \sin H \\ z &= E \sin i' \sin H \end{aligned}\right\} \qquad (1\text{-}1)$$

Hence the direction cosines are : ($\cos H$, $\cos i' \sin H$, $\sin i' \sin H$).

From Figure I-2 find the direction cosines of $\bar{r}$ in the same coordinate system. The angle D may be thought of as the right ascension of the ascending node. The angle G represents the angular distance of $\bar{r}$ from the ascending node. Obviously the angles D and G take on all value in $[0, 2\pi]$.

From Figure I-2 the angle F is the projection of G onto the equatorial plane. Let r' be the projection of $\bar{r}$ onto the same X-Y plane. Then $r' = r \cos$ (angle between $\bar{r}$ and r').

Using Figure I-3 the right spherical triangle with sides G, F and L ' reveals that L ' is the angle between $\bar{r}$ and r'. Then

$$\sin L' = \sin i \sin G \qquad (1\text{-}2)$$

and

$$\begin{aligned} \cos L' &= (1 - \sin^2 L')^{1/2} \\ &= (1 - \sin^2 i \ \sin^2 G)^{1/2} \qquad (1\text{-}3) \end{aligned}$$

Also from the right spherical triangle in Figure I-3:

$$\tan F = \cos i \tan G \qquad (1\text{-}4)$$

hence

$$F = \tan^{-1} (\cos i \tan G) \qquad (1\text{-}5)$$

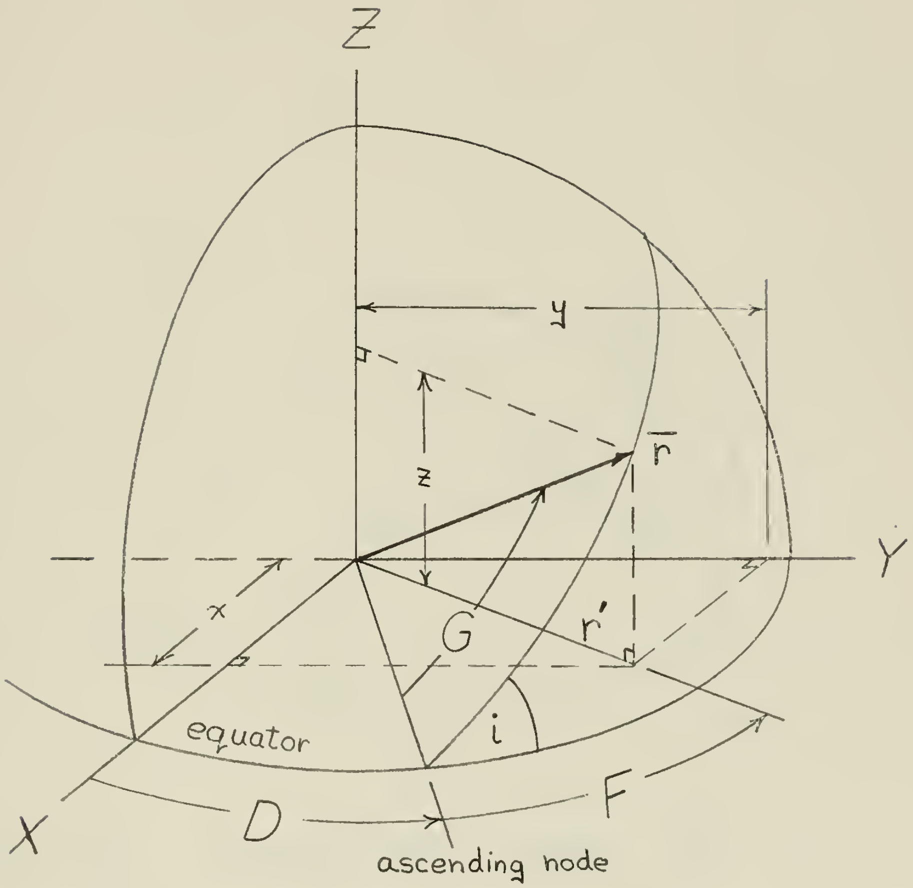

Fig. I-2: Representation of Earth-Satellite vector $\bar{r}$

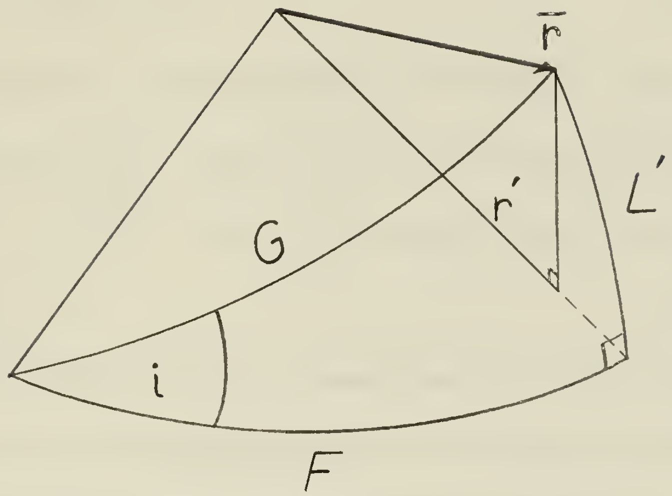

Fig. I-3: Simplified view of Earth-Satellite vector $\bar{r}$

Referring again to Figure I-2,

$$x = r' \cos (D+F)$$

$$= r \cos L' \cos (D+F)$$

$$= r\,(1 - \sin^2 i \sin^2 G)^{1/2} \cos \left[D + \tan^{-1}(\cos i \tan G) \right]; \qquad (1\text{-}6)$$

$$y = r' \sin (D+F)$$

$$= r\,(1 - \sin^2 i \sin^2 G)^{1/2} \sin \left[D + \tan^{-1}(\cos i \tan G) \right]; \qquad (1\text{-}7)$$

$$z = r \sin L'$$

$$= r \sin i \sin G \qquad (1\text{-}8)$$

Using Equations (1-6), (1-7) and (1-8), the direction cosines of $\bar{r}$ are:

$$\Big\{ (1 - \sin^2 i \sin^2 G)^{1/2} \cos \left[D + \tan^{-1}(\cos i \tan G) \right],$$

$$(1 - \sin^2 i \sin^2 G)^{1/2} \sin \left[D + \tan^{-1}(\cos i \tan G) \right],$$

$$\sin i \sin G \Big\} . \qquad (1\text{-}9)$$

Referring to Figure I-4 it is noted that the angle G describes the angular position of the satellite (or the sub-satellite point) in the orbital plane from the ascending node at time t. Hence

$$G = w_s t \qquad (1\text{-}10)$$

where

w_s = angular velocity of satellite in radians/unit time

t = instantaneous time after reference time.

As the satellite proceeds in its orbit to time t, the earth also rotates about its axis with an angular velocity w_e . Then dl is the apparent change in the longitude of the sub-satellite point in time t. Clearly,

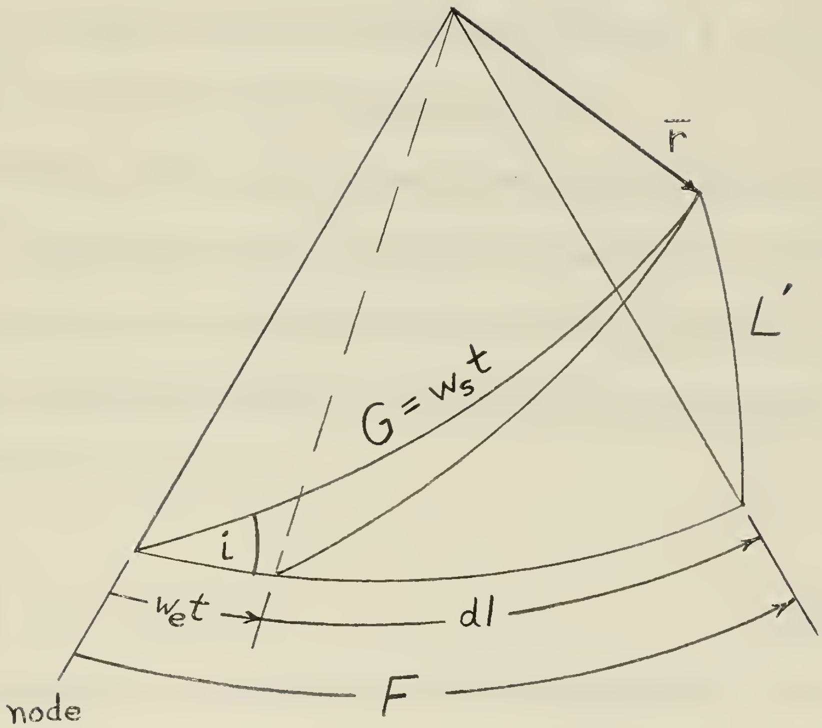

Fig. I-4: Location of sub-satellite point at time t

$$F = w_e t + dl \quad . \tag{1-11}$$

Then the longitude of the sub-satellite point at time t is:

$$\begin{aligned} l'_t &= l'_o + dl \\ &= l'_o + F - w_e t \\ &= l'_o + \tan^{-1}(\cos i \tan G) - w_e t \end{aligned} \tag{1-12}$$

It is more convenient at this point to develop an analytic model which will indicate whether or not the satellite is illuminated by the sun. Figure I-5 depicts the plane containing the $\overline{E}$ and $\overline{r}$ vectors. For simplicity, let both $\overline{E}$ and $\overline{r}$ be unit vectors.

As the satellite orbits about the earth, the angle $\underline{A}$ (between the $\overline{E}$ and $\overline{r}$ vectors) can take on all values in the interval $[0, 2\pi]$. However, since the representation in Figure I-5 is symmetric about the $\overline{E}$ vector, the problem requires an interest in the magnitude of A only in the interval $[0, \pi]$.

Clearly,

$$\overline{E} \cdot \overline{r} = \cos A \tag{1-13}$$

If cos A is non-negative, the satellite will be illuminated by the sun. If cos A is negative, the satellite may or may not be illuminated. Let $\underline{B} = A - \pi/2$ as shown in Figure I-5. Let $\underline{C}$ be the angle at the satellite subtended by the center of the earth and the surface of the earth. Then

$$C = \sin^{-1}\left(\frac{R_e}{R_e + h}\right) \quad , \tag{1-14}$$

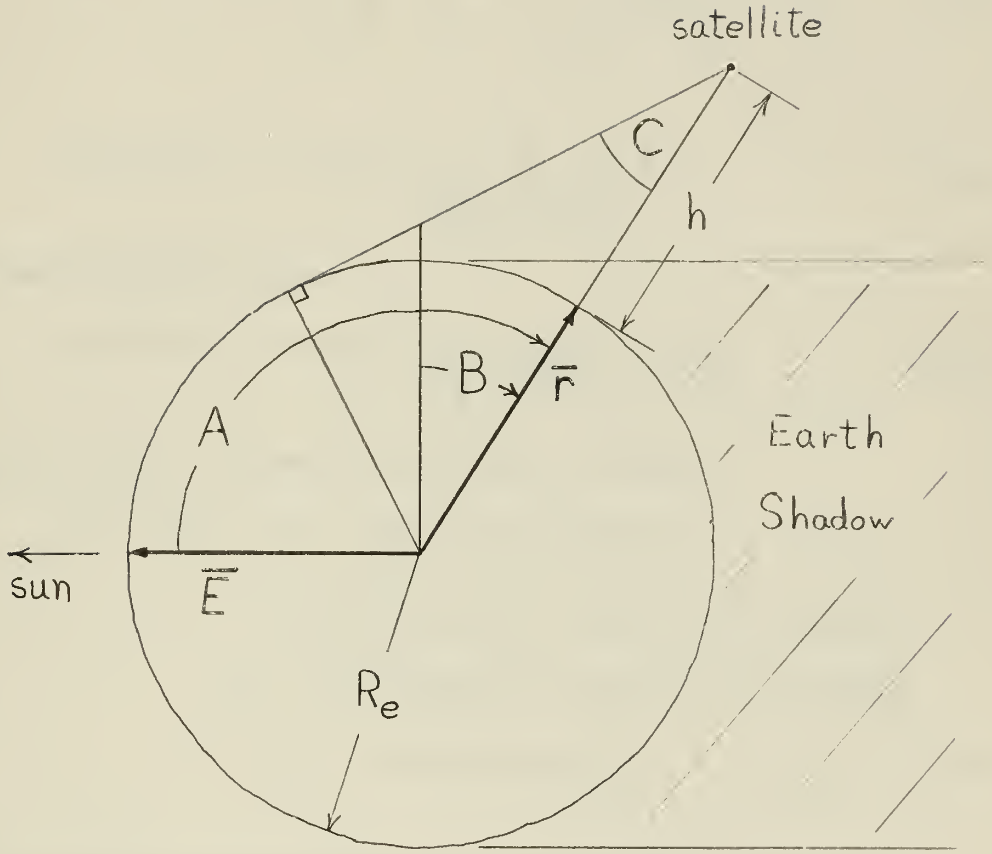

Fig. I-5: Plane containing $\bar{E}$ and $\bar{r}$ vectors

a constant for circular orbits, where R_e is the radius of the earth.

Then if B + C is greater than $\pi/2$, the satellite will be in the shadow cylinder of the earth. Hence if the satellite is illuminated, then

$$\cos^{-1}(\overline{E}\cdot\overline{r}) - \frac{\pi}{2} + \sin^{-1}\left(\frac{R_e}{R_e + h}\right) \leqq \frac{\pi}{2} \qquad (1\text{-}15)$$

or

$$\cos^{-1}(\overline{E}\cdot\overline{r}) + \sin^{-1}\left(\frac{R_e}{R_e + h}\right) \leqq \pi \ . \qquad (1\text{-}16)$$

If it is not illuminated, then

$$\cos^{-1}(\overline{E}\cdot\overline{r}) + \sin^{-1}\left(\frac{R_e}{R_e + h}\right) > \pi \qquad (1\text{-}17)$$

Using the results of Equations (1-1) and 1-9), then:

$$\begin{aligned}\cos A = \overline{E}\cdot\overline{r} = (1 - \sin^2 i \sin^2 G)^{1/2} \times \\ \Big\{\cos H \cos\left[D + \tan^{-1}(\cos i \tan G)\right] + \\ \cos i' \sin H \sin\left[D + \tan^{-1}(\cos i \tan G)\right]\Big\} \\ + \sin i' \sin i \sin H \sin G \qquad (1\text{-}18)\end{aligned}$$

The next stage of the problem is to determine if the satellite track passes within a given angular distance s of either of two geographical points with coordinates (L, W) and (-L, W). With this concept the satellite track may be considered to sweep out a "swath" over the earth's surface, of width 2s. Denote l'_o as the longitude of the ascending node.

For illustrative purposes it is advantageous to show a schematic plot of the satellite track across the surface of the earth, with special emphasis upon its relationship to the designated geographical points. For clarity, a Mercator projection will be used.

For cases where $|i| - s > |L|$, then the situation in Figure I-6 will exist. Clearly, if the value of W lies in one of the intervals $[l_j, l_{j+1}]$ for j = 1, 3, 5, 7, then the desired passage will occur. Indeed this is a necessary condition.

For the case where $|i| - s \leq |L| \leq |i| + s$, then Figure I-7 will apply. Here, the desired passage will occur if W lies in one of the intervals $[l_j, l_{j+1}]$, for j = 1, 3. This too is a necessary condition.

Using these conditions to determine if passage occurs, the time t of closest approach to the geographical point may be evaluated. Using this value of t to compute cos A in Equation (1-18), then the conditions contained in Equation (1-16) are applied to determine if the desired illumination occurs.

It is now necessary to develop expressions for the orbital period and angular velocities for the satellite and the earth. It can be shown that the period of a satellite is:

$$T = \frac{2 \pi (a)^{3/2}}{(GM)^{1/2}} \tag{1-19}$$

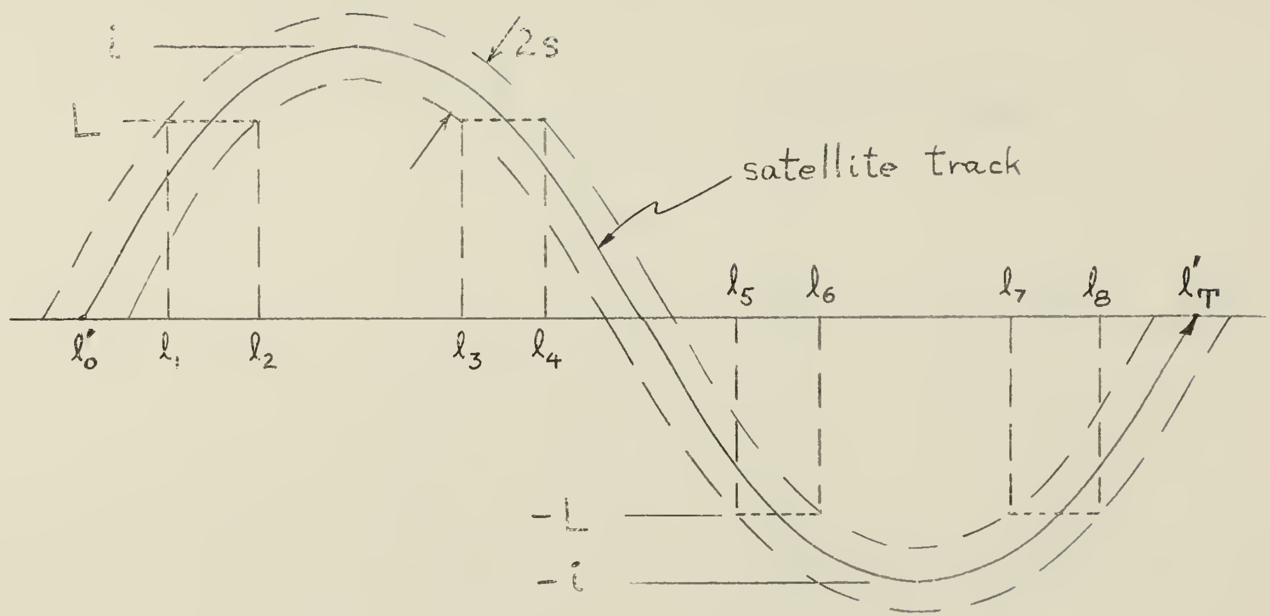

Fig. I-6: Mercator projection of track; $|i| - s > |L|$

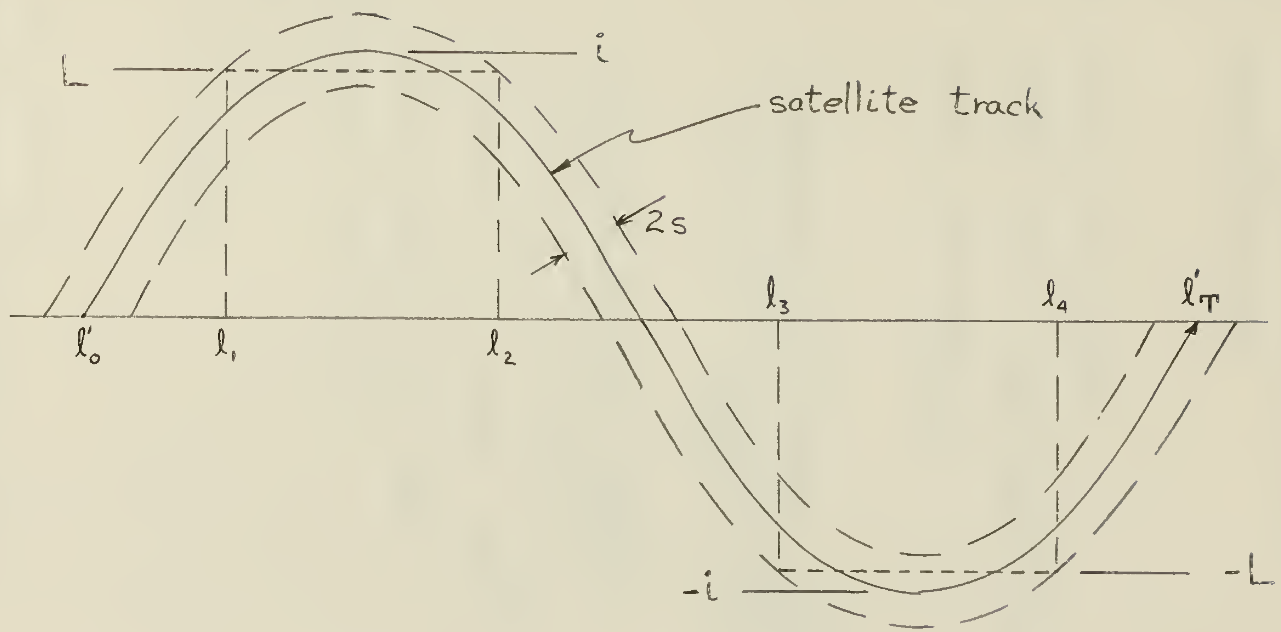

Fig. I-7: Mercator projection of track; $|i| - s \leq |L| \leq |i| + s$

where

a = semi-major axis of the orbital ellipse

= $R_e + h$, for circular orbits

G' = universal gravitational constant

M' = mass of the earth.

In mks units,

$R_e = 6.371 \times 10^6$ meters

$G' = 6.668 \times 10^{-11}$ newton-meter^2/kilogram^2

$M' = 5.98 \times 10^{24}$ kilograms

Then

$$T = \frac{2\pi\,(6.371 \times 10^6 \text{ meters} + h)^{3/2}}{\left(6.668 \times 10^{-11}\ \dfrac{\text{nt.}-\text{m}^2}{\text{kg}^2} \times 5.98 \times 10^{24}\text{ kg.}\right)^{1/2}}$$

$$= \frac{2\pi\,(6.371 \times 10^6 \text{ m.} + h)^{3/2}}{(3.99 \times 10^{14}\ \text{m}^3/\text{sec}^2)^{1/2}}$$

$$= 3.148 \times 10^{-7}\,(6.371 \times 10^6 + h)^{3/2} \text{ sec}$$

$$= 5.246 \times (6.371 + h)^{3/2} \text{ min.} \qquad (1\text{-}20)$$

where h is given in units of 10^6 meters.

Since $w_s T = 2\pi$,

$$w_s = 2\pi/T \qquad (1\text{-}21)$$

and

$$w_e = 2\pi/T_e$$

where T_e is the rotational period of the earth and

$$w_e = (2\pi / 1440)\ \text{min.}$$

$$= 4.363 \times 10^{-3}\ \text{min.} \qquad (1\text{-}22)$$

CHAPTER II

PROBABILITY FORMULATION

The next stage leading to a solution of this problem is the formulation of a probability statement. The probability sought is the probability that illumination occurs simultaneously with the desired passage by the n^{th} orbit, for $n = 1, 2, \ldots, 16$.

Looking at the first orbit, the longitude system may, without any loss of generality, be arbitrarily defined such that the longitude of the ascending node, l'_o, is equal to zero, and all other longitudes are measured eastward from zero through a total angle of 2π.

Let M be the event that the desired passage occurs sometime during the first orbit, and let N be the event that the desired illumination occurs at the particular time t when the satellite makes its closest approach to the geographical point. Note that M is a time-independent event, while N is strictly dependent upon the value of t. The probability sought is $P\left[MN\right]$.

Studying each event separately, it is clear that

$$P\left[M\right] = P\left[l_j \leqq W \leqq l_{j+1}\right] \qquad (2\text{-}1)$$

where $j = 1$ and 3, or else $j = 1, 3, 5, 7$, depending upon the values of i, s and L.

This equation can be written in more meaningful language if $P\left[M_j\right]$ is defined as the probability that W lies in $\left[l_j, \; l_{j+1}\right]$.

Clearly, the events M_1, M_3, . . . , are mutually exclusive, hence

$$P\left[M\right] = \sum_{\text{odd } j} P\left[M_j\right] \qquad (2\text{-}2)$$

Also, from Equation (1-16),

$$P\left[N\right] = P\left[A + C \leqq \pi\right] = P\left[A \leqq \pi - C\right] \qquad (2\text{-}3)$$

Hence the random variable associated with event M is W, and it appears intuitively correct that A is the random variable associated with event N. Then the joint probability sought is

$$P\left[MN\right] = \sum_{\text{odd } j} P\left[M_j N\right] \qquad (2\text{-}4)$$

To verify that A is a random variable, note that Equation (1-18) shows that A is a function of i, G, D and H. But for a particular time t, the values of i and G are constant. Then $P\left[A \leqq \pi - C\right]$ is simply the probability that D and H take on a range of values such that $A \leqq \pi - C$, which implies that D and H are random variables with associated probability density functions, $f_D(d)$ and $f_H(h)$, both defined on the interval $\left[0, 2\pi\right]$. Care should be taken to avoid confusion with h, the satellite altitude; however, in most situations the meaning will be clear from the context.

Using the basic assumptions and the physical interpretations of D and H, it appears intuitively correct to state that they are independent, uniformly distributed random variables such that

$$f_D(d) = \begin{cases} 1/2\pi & \text{for } 0 \leq d \leq 2\pi \\ 0 & \text{elsewhere} \end{cases} \tag{2-5}$$

$$f_H(h) = \begin{cases} 1/2\pi & \text{for } 0 \leq h \leq 2\pi \\ 0 & \text{elsewhere} \end{cases} \tag{2-6}$$

It quickly follows that $A = A(G,H)$ is a random variable with an associated probability density function, $f_A(a)$, defined over the interval $\left[0, \pi\right]$.

While it appears unlikely that the mathematical expression for $f_A(a)$ could be derived quickly by analytic methods, the formulation of the problem will continue as if $f_A(a)$ were completely described.

Before proceeding further with Equation (2-4) it is necessary to develop the form of the probability density function for W. Since the only quantity of interest is $W - l'_0$, and since one of the basic initial assumptions was that l'_o was uniformly distributed, it follows that W is also uniformly distributed. Arbitrarily setting l'_o equal to zero, as before, merely ensures that W is defined over the interval $\left[0, 2\pi\right]$.

Then

$$f_W(w) = \begin{cases} 1/2\pi & \text{for } 0 \leq w \leq 2\pi \\ 0 & \text{elsewhere} \end{cases} \tag{2-7}$$

Now by the rules for conditional probabilities,

$$P\left[M_j N\right] = P\left[N \middle| M_j\right] \cdot P\left[M_j\right] \tag{2-8}$$

and

$$P\left[MN\right] = \sum_{\text{odd } j} P\left[N \middle| M_j\right] \cdot P\left[M_j\right] \qquad (2\text{-}9)$$

Assume now that, associated with $P\left[N \middle| M_j\right]$, there exists a conditional probability density function $f_{A|W}(a|w)$, defined over the interval $\left[0, \pi\right]$. Then Equation (2-9) may be written as

$$P\left[MN\right] = \sum_{\text{odd } j} \int_{l_j}^{l_{j+1}} \int_{0}^{\pi - C} f_{A|W}(a|w)\, f_W(w)\, da\, dw$$

$$= \frac{1}{2\pi} \sum_{\text{odd } j} \int_{l_j}^{l_{j+1}} \int_{0}^{\pi - C} f_{A|W}(a|w)\, da\, dw \qquad (2\text{-}10)$$

using the results of Equation (2-7).

The final probability to be calculated is the joint cumulative probability that the desired events M and N will occur simultaneously at least once by the n^{th} orbit, for $n = 1, 2, \ldots, 16$. To accomplish this requires some new notation.

Let P_n represent the desired probability of at least one such joint occurrence by the n^{th} orbit. Let P(1) be the probability that the desired event occurs on the first orbit; clearly P(1) is the solution to Equation (2-10). Let $P(n', n+1)$ be the joint probability of no such occurrence on the first n orbits and one occurrence on the $(n+1)^{st}$ orbit.

Then

$$P_1 = P(1)$$

$$P_2 = P_1 + P(1', 2)$$

$$P_3 = P_2 + P(2', 3)$$

and, in general,

$$P_{n+1} = P_n + P(n', n+1) \quad . \tag{2-11}$$

This equation describes symbolically the procedure for calculating P_n for $n = 1, 2, \ldots, 16$. The problem of calculating the probabilities for orbits subsequent to the first can be considerably alleviated by utilizing a characteristic of the satellite tracks.

From Equation (1-12) it is easily shown that the longitude of the ascending node for the second orbit is

$$l'_T = l'_o - w_e T \quad , \tag{2-12}$$

i.e., l'_T has been displaced westward of l'_o by the amount $w_e T$. But this is equivalent to retaining l'_o as the longitude of the ascending node for subsequent orbits and displacing W eastward a distance $w_e T$. In other words, Equation (2-10) may be used for the n^{th} orbit, providing the longitude of the geographical points is set equal to

$$(W)_n = W + (n - 1) w_e T \quad \text{for } n = 1, 2, \ldots, 16 \tag{2-13}$$

Nonetheless, there remains unsolved the problems of determining the limits of integration (with respect to w) and discovering the analytic expression for $f_{A|W}(a|w)$ in Equation (2-10). These problems

will be avoided by abandoning the analytic approach in favor of a computer simulation program using Monte Carlo methods.

CHAPTER III

SIMULATION PROBLEM

It is expedient first to examine the form of the desired solution in order to assess the proper method of arriving at it.

The cumulative probability distribution function sought is not a continuous function, but a monotonic step function, with certain probabilities associated with each point corresponding to the end of an orbital period. In fact, the desired result will be a distribution function truncated at 16 orbital periods.

Now assume for the moment that the problem can be set up for simulation, where a suitable number, k, of samples can be obtained from a pseudo-random number generator. Let

x_1 = number of times that passage occurs on the 1^{st} orbit

x_i = number of times that passage occurs on the i^{th} orbit and not on the i-1 previous orbits, for i = 2,3,... 16.

y_1 = number of times that illumination occurs on the 1^{st} orbit, given that simultaneous passage occurs on the first orbit.

y_i = number of times that illumination occurs on the i^{th} orbit, given simultaneous passage on the same orbit, and that such joint event did not occur on the i-1 previous orbits, for i = 2,3,... 16.

Here, "passage" implies that the satellite track passes within the

specified angular distance s of either of the two geographical points.

With these precise definitions it is possible to assert that:

$$E\left[P_1\right] = E\left[P(1)\right] \cong \frac{y_1}{k}$$

$$E\left[P_2\right] \cong \frac{y_1}{k} + E\left[P(1',2)\right] \cong \frac{y_1 + y_2}{k}$$

and, in general,

$$E\left[P_n\right] \cong \frac{1}{k} \sum_{i=1}^{n} y_i \qquad (3-1)$$

If $P_n(p)$ is defined to be the cumulative probability of passage (independent of illumination), then, in similar manner,

$$E\left[P_n(p)\right] \cong \frac{1}{k} \sum_{i=1}^{n} x_i \qquad (3-2)$$

Now define u_{ij} and v_{ij} such that

$$u_{ij} = \begin{cases} 1 & \text{if the desired passage occurs for the first time on the } i^{th} \text{ orbit, for the } j^{th} \text{ sample random number value} \\ 0 & \text{Otherwise} \end{cases} \qquad (3-3)$$

$$v_{ij} = \begin{cases} 1 & \text{If the desired illumination and passage occurs for the first time on the } i^{th} \text{ orbit, for the } j^{th} \text{ sample random number value.} \\ 0 & \text{Otherwise} \end{cases} \qquad (3-4)$$

where $j = 1, 2, \ldots, k$. Note that $v_{ij} = 1$ does not require that $u_{ij} = 1$.

Then

$$x_i = \sum_{j=1}^{k} u_{ij} \tag{3-5}$$

and

$$y_i = \sum_{j=1}^{k} v_{ij} \tag{3-6}$$

Hence

$$E\left[P_n\right] \cong \frac{1}{k} \sum_{i=1}^{n} \sum_{j=1}^{k} v_{ij} \tag{3-7}$$

and

$$E\left[P_n(p)\right] \cong \frac{1}{k} \sum_{i=1}^{n} \sum_{j=1}^{k} u_{ij} \tag{3-8}$$

Equations (3-7) and (3-8) can be considered as the basis for the simulation model.

The design of the computer simulation program is described in some detail in Appendix A, with a description of some of the procedures used to reduce the required computation time. The language used in writing this program was the CDC-1604 version of NELIAC, which is peculiarly well-adapted to this type of program.

Appendix B contains a summary of the output of this program, along with a sensitivity analysis for the effect of single parametric variations of satellite altitude, orbital inclination, latitude of geographical points, and angular distance s.

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4. The NELIAC Compiler Language, Department of Operations Research, U.S. Naval Postgraduate School, Monterey, Calif., 1963.

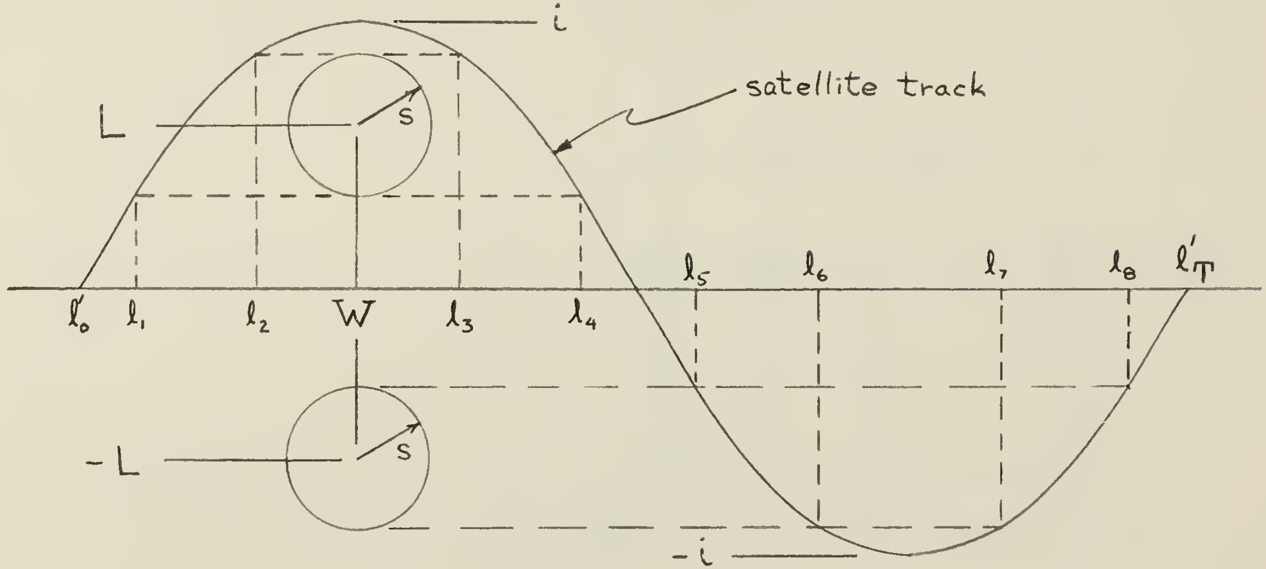


Fig. A-1: Mercator projection of track; $|i| - s > |L|$

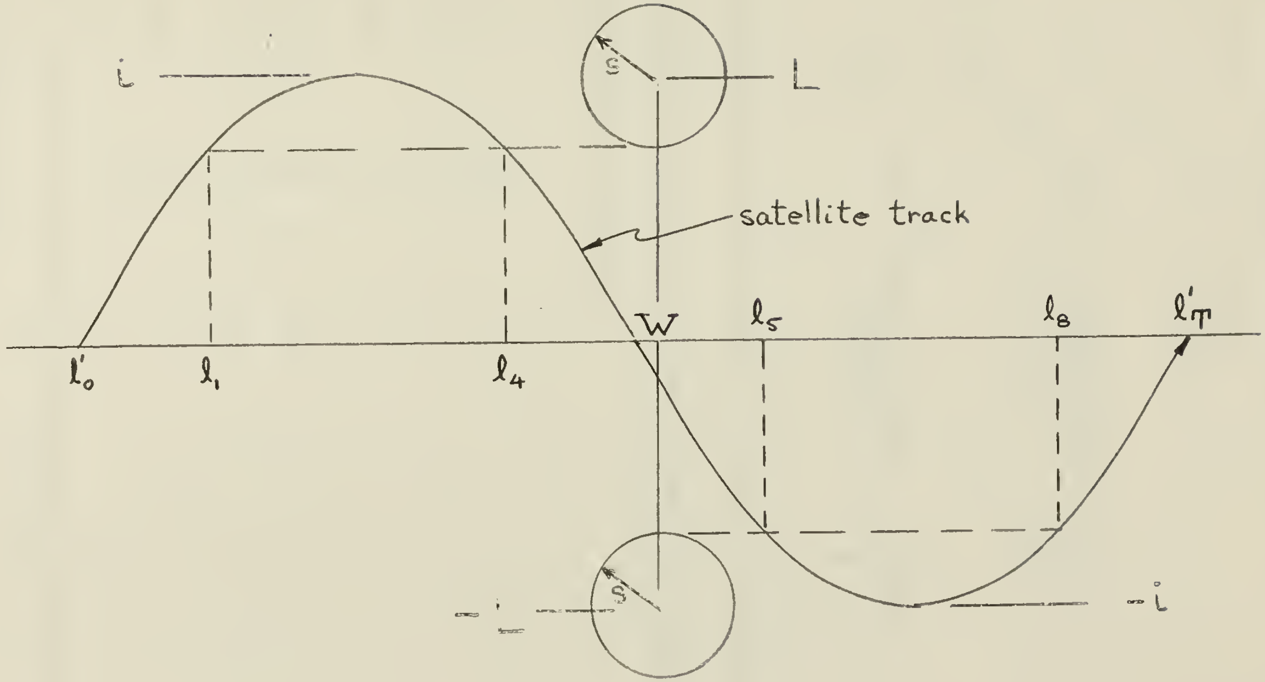


Fig. A-2: Mercator projection of track; $|i| - s \leqq |L| \leqq |i| + s$

clearly, passage will occur if l'_t and W lie within this interval. The numbering of the subscript j is designed to facilitate the computer programming.

From Equations (1-2) and (1-10):

$$\sin L' = \sin i \sin w_s t$$

hence for $L' = L - s$,

$$t = \frac{1}{w_s} \sin^{-1} \left(\frac{\sin (L - s)}{\sin i} \right) = t_1 \qquad \text{(A-1)}$$

and for $L'_t = L + s$,

$$t = \frac{1}{w_s} \sin^{-1} \left(\frac{\sin (L + s)}{\sin i} \right) = t_2 \qquad \text{(A-2)}$$

where t_j corresponds to the time at which the satellite track attains a value of longitude equal to l_j. The remaining values of t_j for j 2 can be determined quickly by

$$t_3 = \frac{T}{2} - t_2$$

$$t_4 = \frac{T}{2} - t_1$$

$$t_5 = \frac{T}{2} + t_1$$

$$t_6 = \frac{T}{2} + t_2$$

$$t_7 = \frac{T}{2} + t_3$$

$$t_8 = \frac{T}{2} + t_4$$

which can be easily verified by the symmetry of the track. T is of course the satellite period.

Using Equation (1-12) it follows that

$$l_j = \tan^{-1}(\cos i \tan w_s t_j) - w_e t_j$$

and accordingly

$$l_1 = \tan^{-1}(\cos i \tan w_s t_1) - w_e t_1 \tag{A-3}$$

$$l_2 = \tan^{-1}(\cos i \tan w_s t_2) - w_e t_2 \tag{A-4}$$

where l_1 and l_2 are defined as in Figures A-1 and A-2. In these figures if W lies within any of the intervals depicted then passage will occur. The remaining values of l_j can be computed in exactly the same manner as Equations (A-3) and (A-4).

Then the first test for the desired passage is determining if the sample value for the random variable W lies within one of the aforementioned intervals. If W satisfies this test, then the test for illumination may be made. However, it is possible for W to be outside such intervals and still provide for the desired passage. If W lies outside of, but sufficiently close to, either end point of the depicted intervals, then passage may occur as in Figure A-3.

At this point it is appropriate to introduce the approximation $\underline{s}'$ of the true instantaneous angular distance between the sub-satellite point and the designated geographical point. For values of s' sufficiently small, the curved surface of the earth can be approximated by a plane. Then, from Figure A-3,

$$s' = \left[\left| L - L'_t \right|^2 + \left| W - l'_t \right|^2 \cos^2 L'_t \right]^{1/2} \tag{A-5}$$

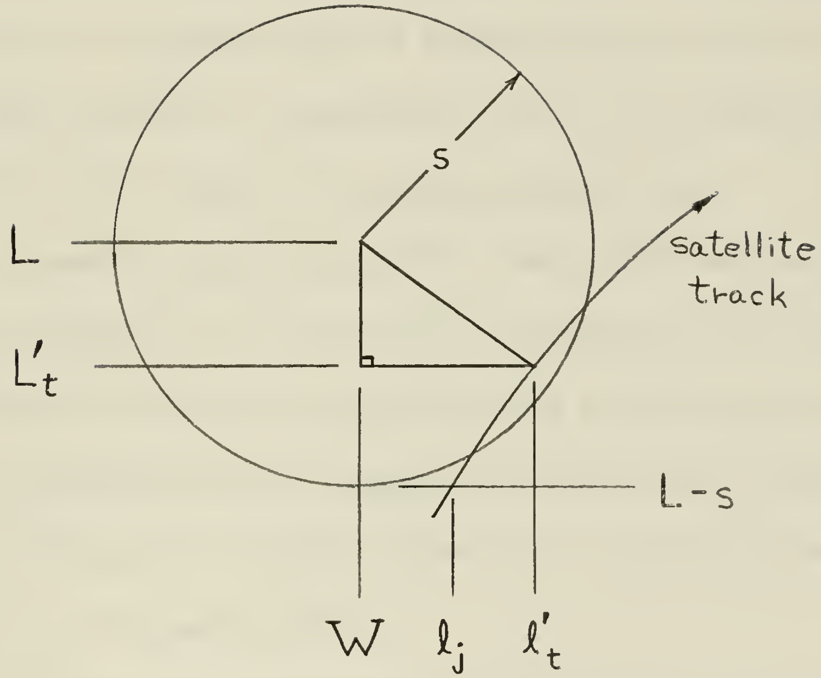


Fig. A-3: Mercator projection of track; special case

where the cosine term is used to compensate for the "shrinkage" in distance between adjacent meridians of longitude with increasing latitude.

It should be obvious that passage cannot occur if $\left| (W - l'_t) \times \cos L'_t \right| \geq s$, hence this quantity may be used as an upper bound in an iterative procedure for computing successive values of s' for successive values of time. Due to the monotonic nature of the satellite track, a minimum value of s' will exist for some value of t, say t*. This value of s' is compared to s; if $s' \leq s$, then the desired passage will occur. Otherwise, passage cannot occur.

If these tests are applied to the i^{th} orbit and j^{th} sample value of the random variable W, the value of u_{ij} in Equation (3-3) is determined. If $u_{ij} = 1$, then the illumination condition may be determined by application of Equation (1-16). Applying this test to the i^{th} orbit and j^{th} sample values of the random variables W, D and H determines the value of v_{ij} in Equation (3-4).

However, it may happen that the desired passage will occur with no illumination on an earlier orbit, but with the desired passage and illumination on a later orbit. In order to compute $P_n(p)$, it is necessary to record the earlier event where $u_{ij} = 1$, but continuing the process in order to compute P_n.

Since this simulation program was only incidental to the solution of the main problem, the computer flow diagrams and NELIAC

language flow charts have not been included. However, the IBM binary card decks and program print-out are available from Professor Richard M. Thatcher, of the Operations Analysis Department.

APPENDIX B

SUMMARY OF PROGRAM OUTPUT

The parametric inputs to the simulation program were satellite altitude h, orbital inclination i, latitude of designated geographical points L, and angular distance from these points s. The values of h and s are given in units of nautical miles, but were converted internally to units of 10^6 meters and radians respectively; the values of i and L are given in degrees, but were likewise converted internally to radians. The values of these parameters and the incremental values used in the program are given below:

h:	100(300)1900 nautical miles
i:	$20^0(8^\circ)68^\circ$
L:	$25^\circ(10^\circ)65^\circ$
s:	100(100)500 nautical miles

The value of k (sample size) was hopefully selected to be 200, in order that the approximations contained in Chapter III might be statistically valid, and yet require a reasonably short amount of computer time. Even so, about three and one-half hours actual operating time was consumed on the computer for a single run through all values of the above parameters.

One immediate result from the output of this program was the discovery of a "semi-optimal" distance s = 300 nautical miles.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

In many instances, for various values of h, i and L, the cumulative probabilities revealed little or no increase with values of s greater than 300. In a significant number of cases the rate of increase in probability declined sharply after s = 300; it is to be noted, however, that this was not always the case. Frequently the rate of increase in probability with increasing s was approximately linear. Nevertheless, it was decided to limit the tabulated values of the probabilities to those computed for s = 300. To have done otherwise would have increased the size of this appendix by a factor of five.

Another unexpected discovery concerned certain substantial decreases in cumulative probabilities for h = 1900 nautical miles. The reduction did not occur consistently for all values of the other parameters, however. It is suspected that this reduction is most likely a statistical anomaly; unfortunately insufficient time was available for further repeated runs using this value of h. This so-called anomaly is suggested as a topic for further study, as is an analysis of variance for repeated runs. The computer program is easily adapted to the study of the effects of variation of any one parameter.

Figures B-1 through B-10 give the joint probability of passage and illumination vs. number of orbital periods, while Figures B-11 through B-15 give the independent cumulative probability of passage vs. number of orbital periods.

Figures B-1 through B-4 and B-11 through B-15 represent probabilities averaged for all the altitudes examined and, as such, show the sensitivity due to variations in inclination. The rationale for this averaging process lies in the fact that the variations in probabilities due to increasing altitude were sufficiently small to warrant doing so.

Figures B-5 through B-10, however, are given for specific latitudes and inclinations, and show the sensitivities due to variations in altitude. In these cases it will be noted that the values of L and i are approximately the same. In these cases the probabilities are consistently higher than in cases where i is considerably greater than L. Of course, when i is less than L by 3° or more, no passage occurs.

These graphs have been smoothed to some extent over the raw statistical output, but the general trend has been retained in all cases. No attempt should be made to interpolate between discrete orbital periods.

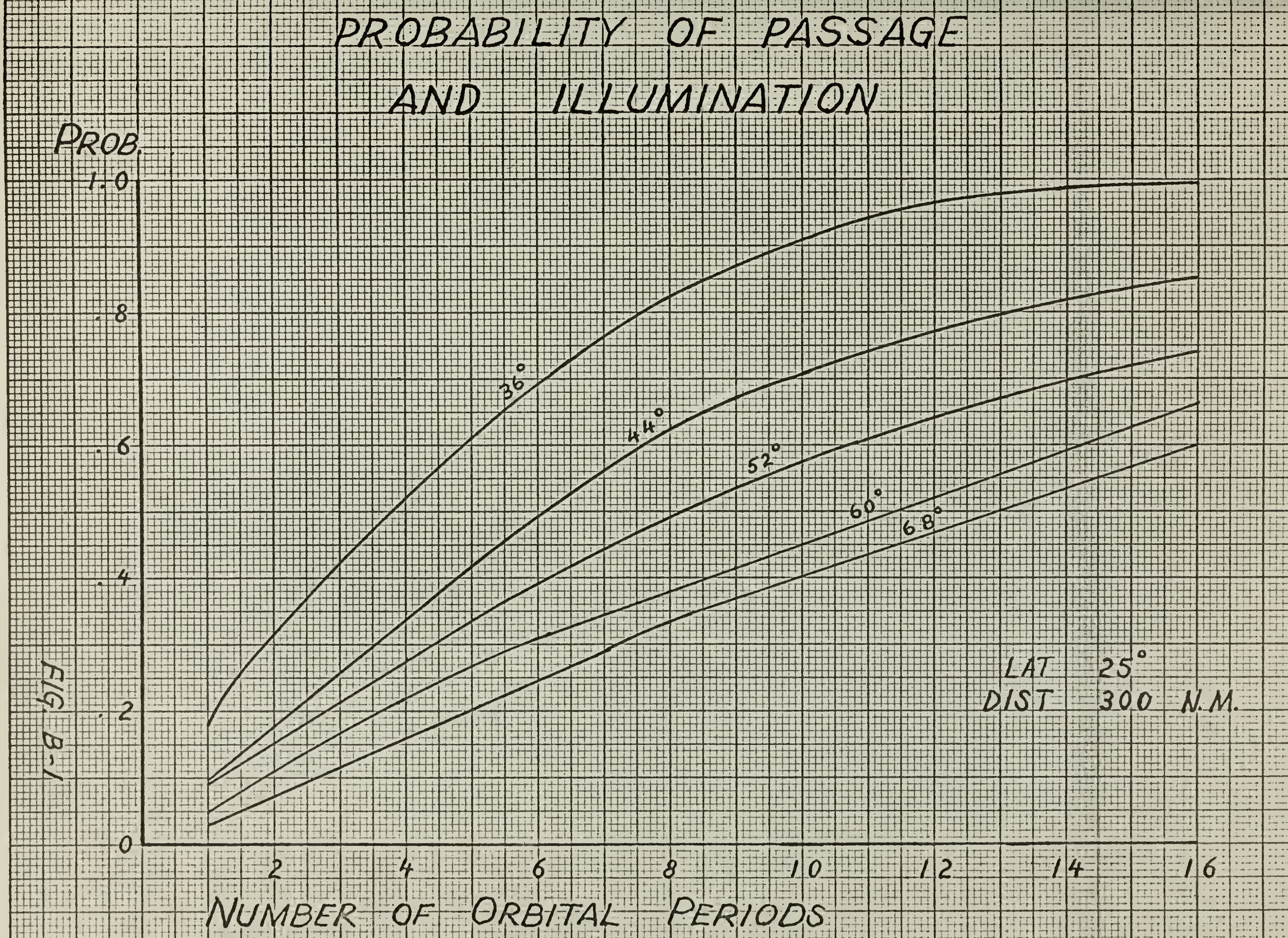

PROBABILITY OF PASSAGE
AND ILLUMINATION
PROB.
1.0
.8
.6
.4
.2
0
2
4
6
8
10
12
14
16
NUMBER OF ORBITAL PERIODS
36°
44°
52°
60°
68°
LAT 25°
DIST 300 N.M.

FIG. B-1

PROBABILITY OF PASSAGE AND ILLUMINATION

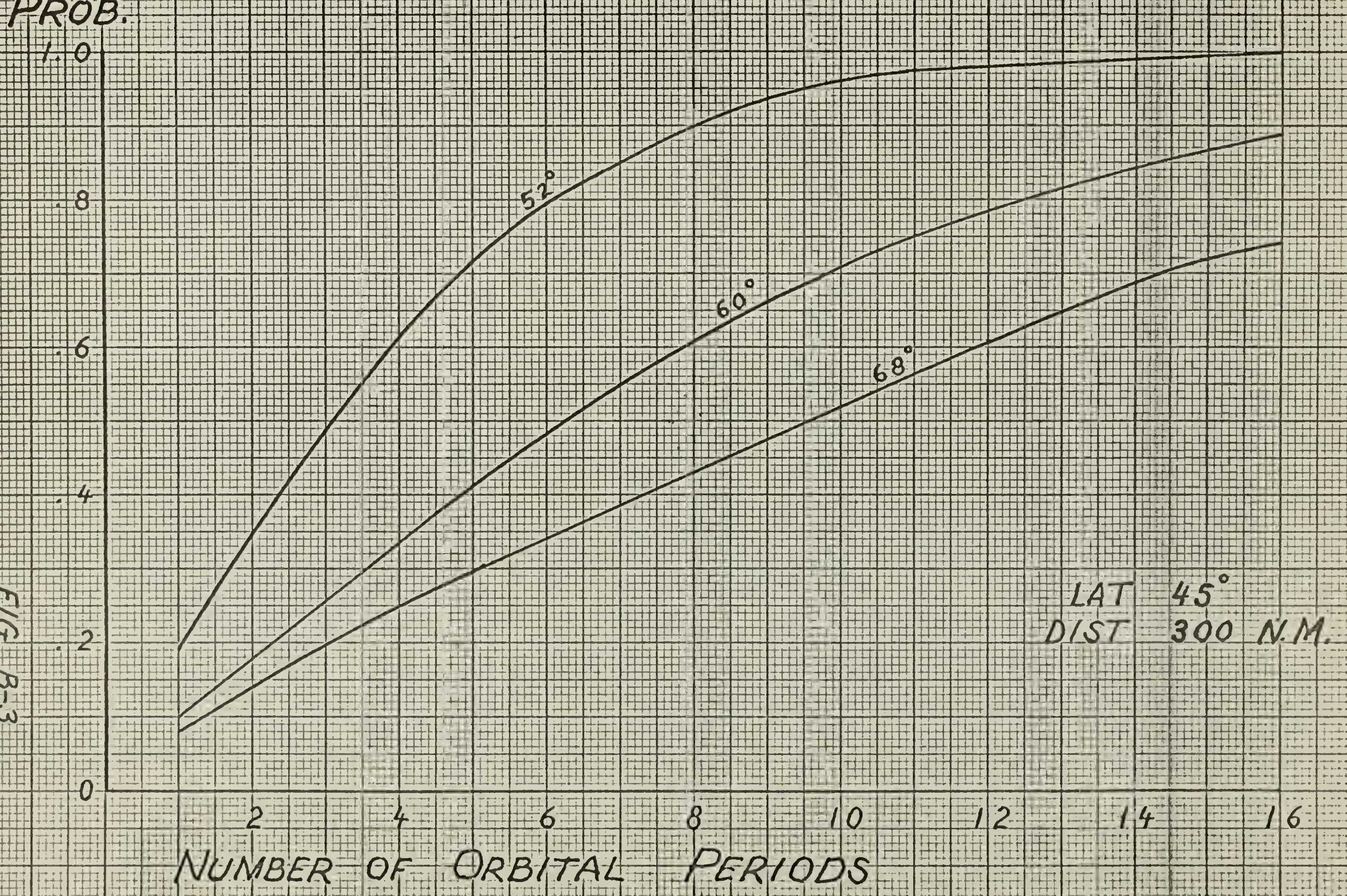


FIG. B-3

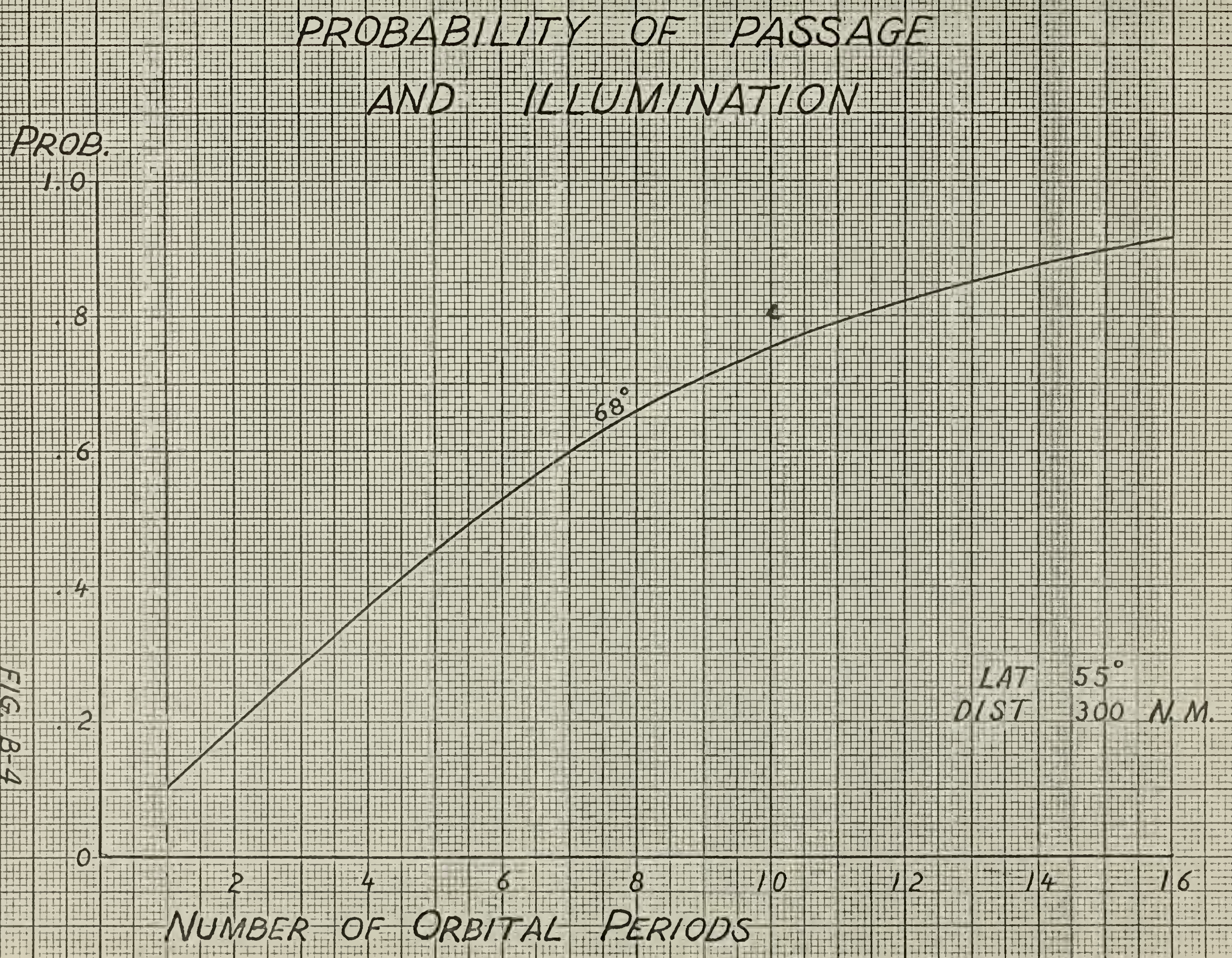

PROBABILITY OF PASSAGE
AND ILLUMINATION
PROB.
1.0
.8
.6
.4
.2
0
68°
LAT 55°
DIST 300 N.M.
2
4
6
8
10
12
14
16
NUMBER OF ORBITAL PERIODS

FIG. B-4

PROBABILITY OF PASSAGE AND ILLUMINATION

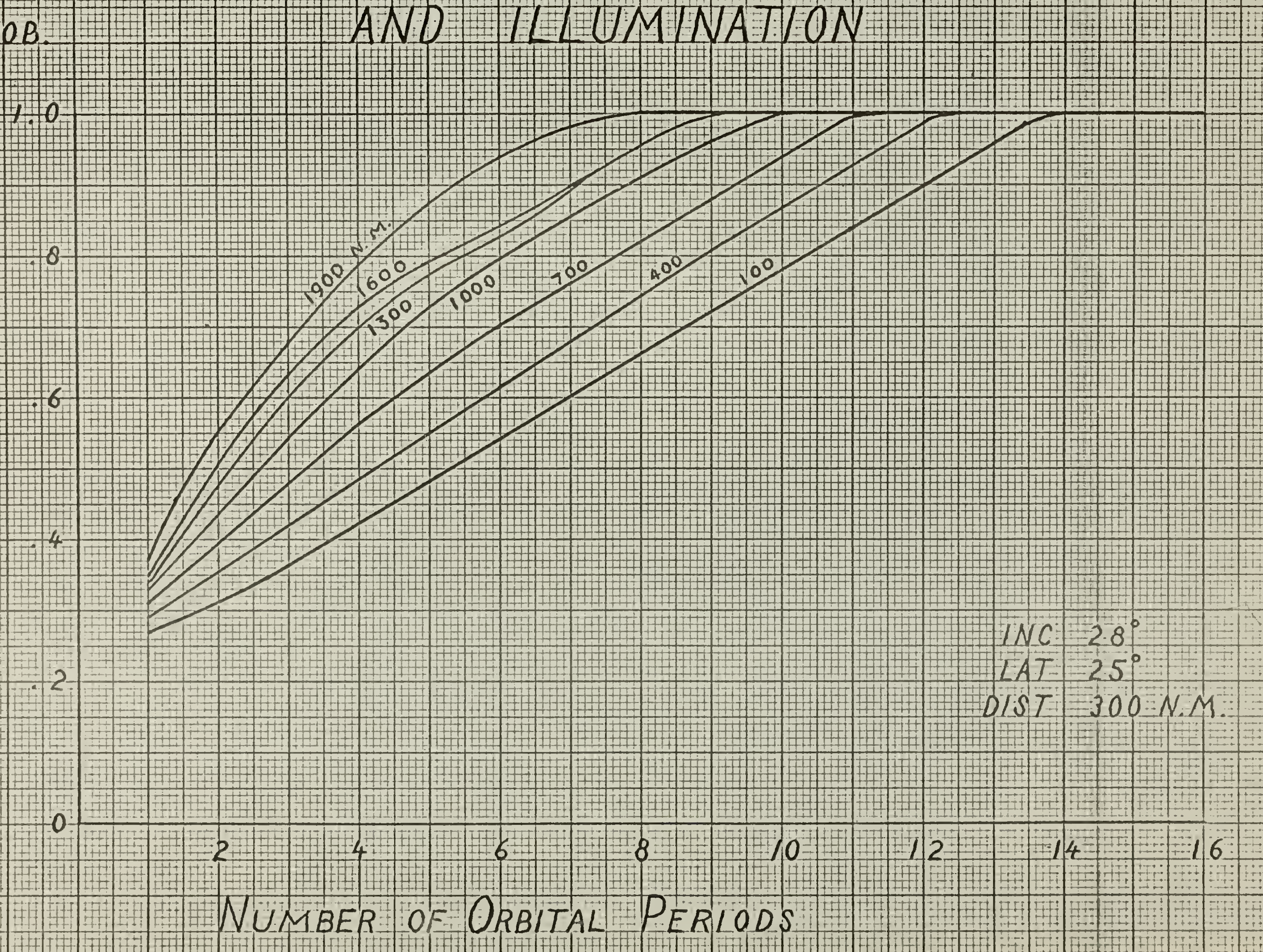


FIG. B-5

PROBABILITY OF PASSAGE AND ILLUMINATION

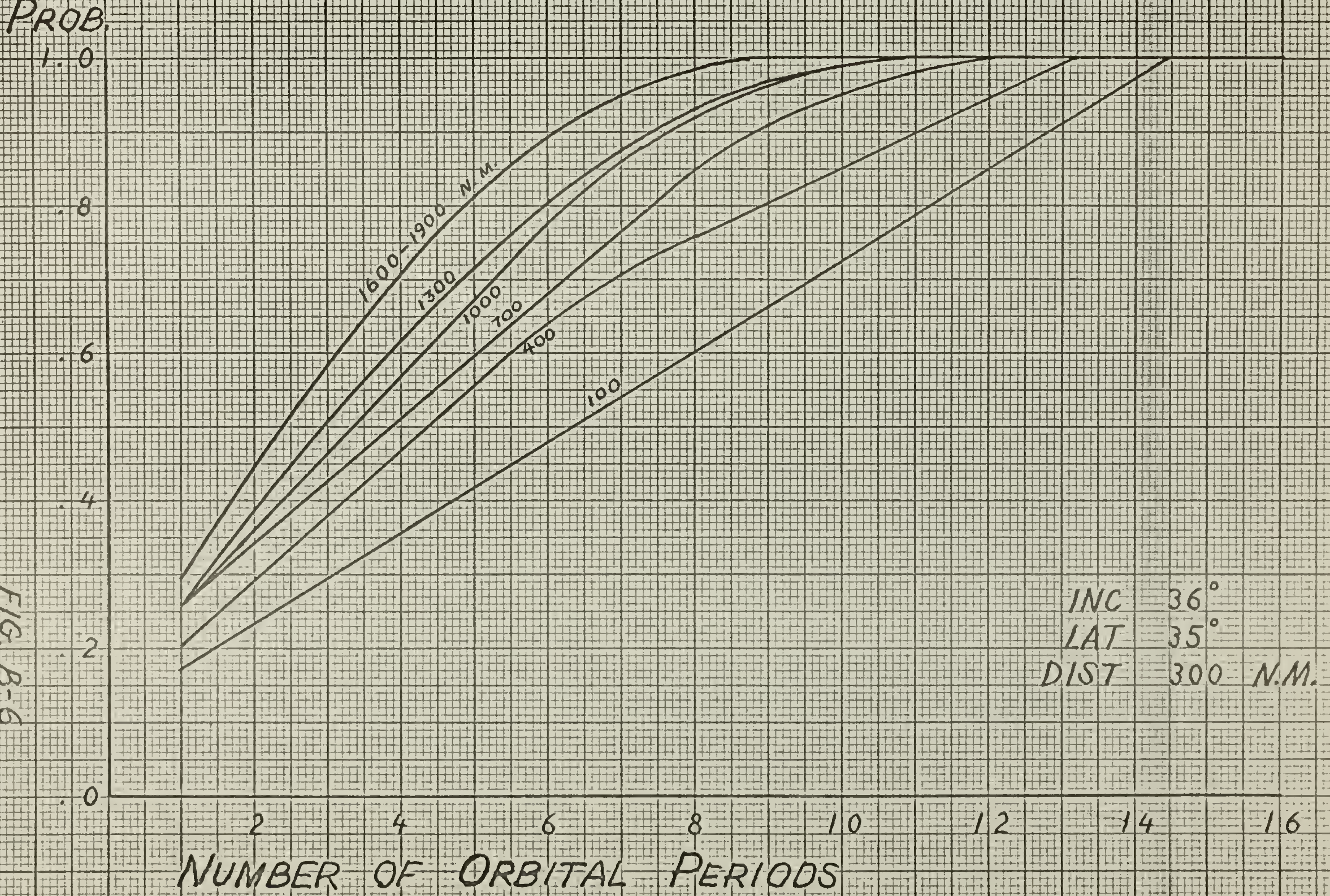


FIG. B-6

PROBABILITY OF PASSAGE AND ILLUMINATION

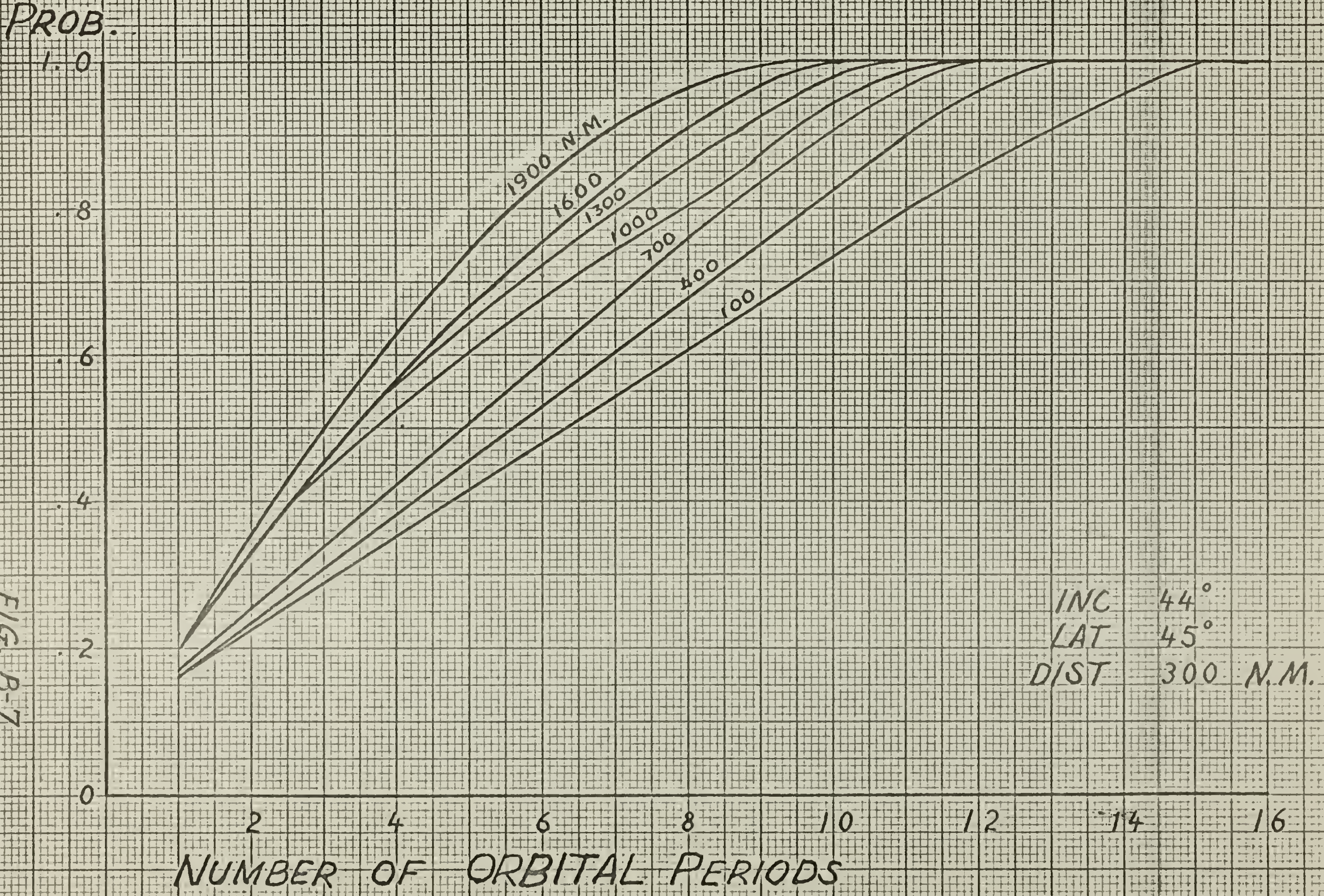
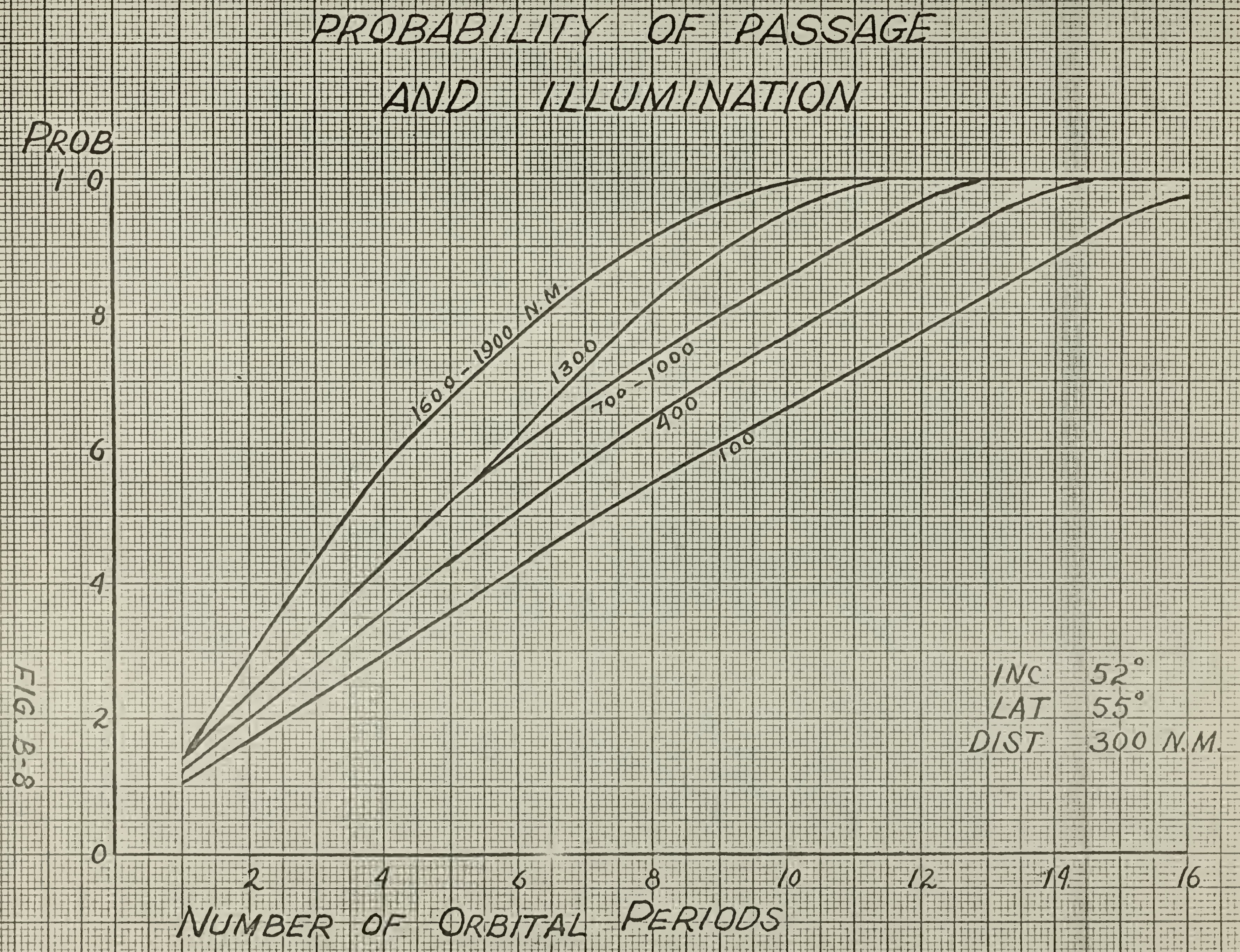


FIG. B-7



PROBABILITY OF PASSAGE
AND ILLUMINATION
PROB
1.0
8
6
4
2
0
2
4
6
8
10
12
14
16
NUMBER OF ORBITAL PERIODS
1600-1900 N.M.
1300
700-1000
400
100
INC 52°
LAT 55°
DIST 300 N.M.

FIG. B-8

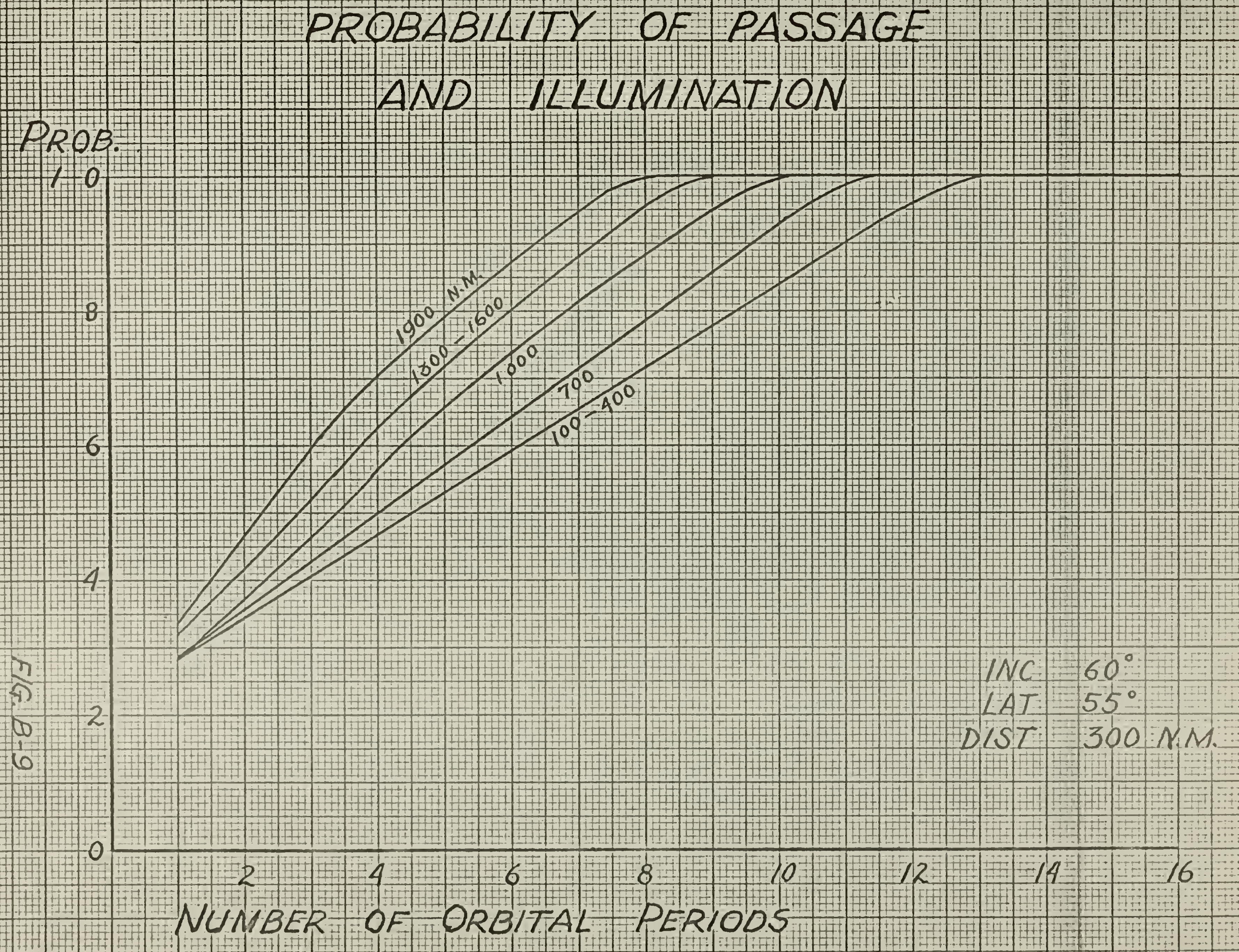

PROBABILITY OF PASSAGE
AND ILLUMINATION
PROB.
1.0
.8
.6
.4
.2
0
2
4
6
8
10
12
14
16
NUMBER OF ORBITAL PERIODS
1900 N.M.
1300 - 1600
1000
700
100 - 400
INC 60°
LAT 55°
DIST 300 N.M.

FIG. B-9

PROBABILITY OF PASSAGE

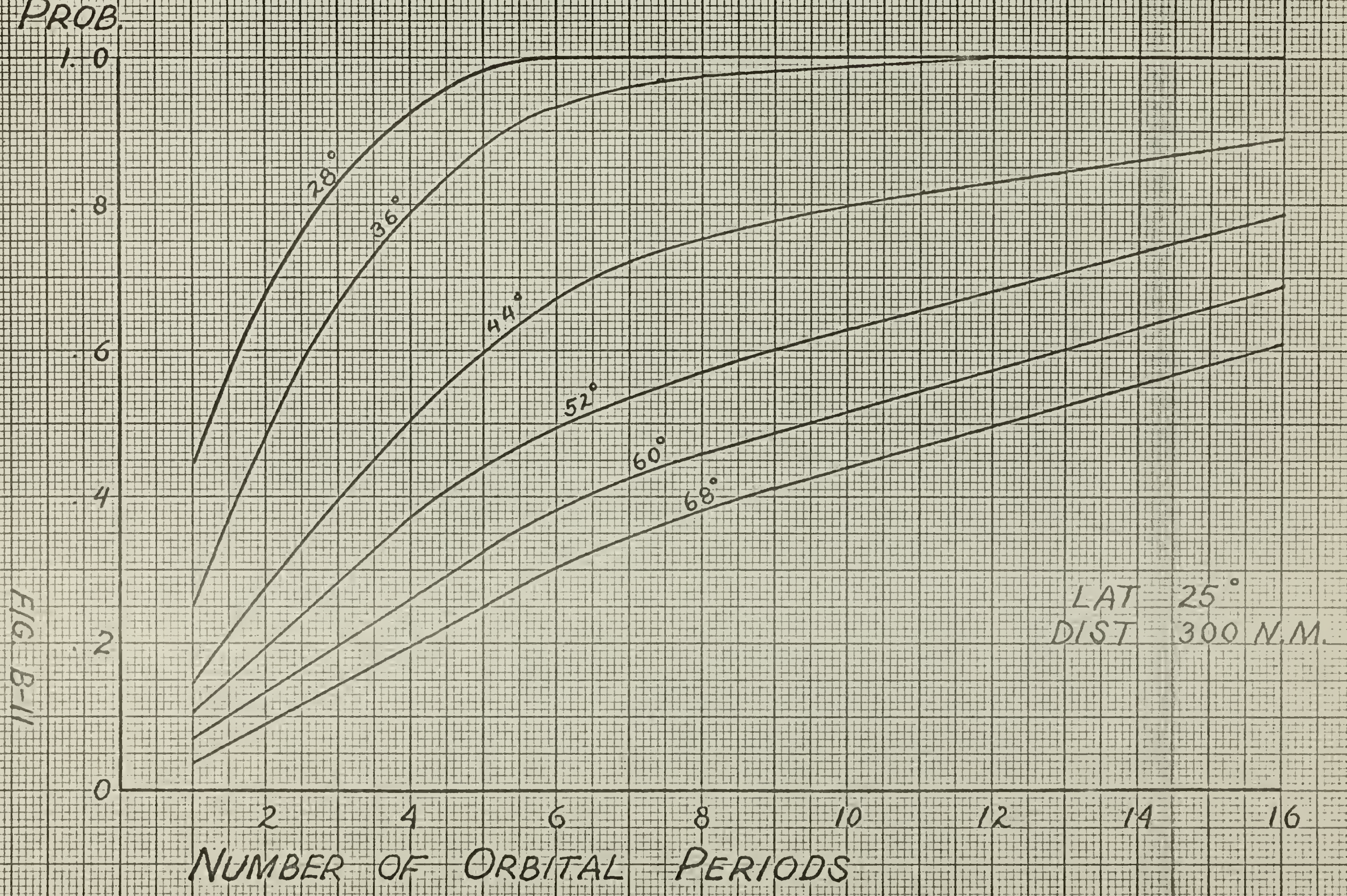
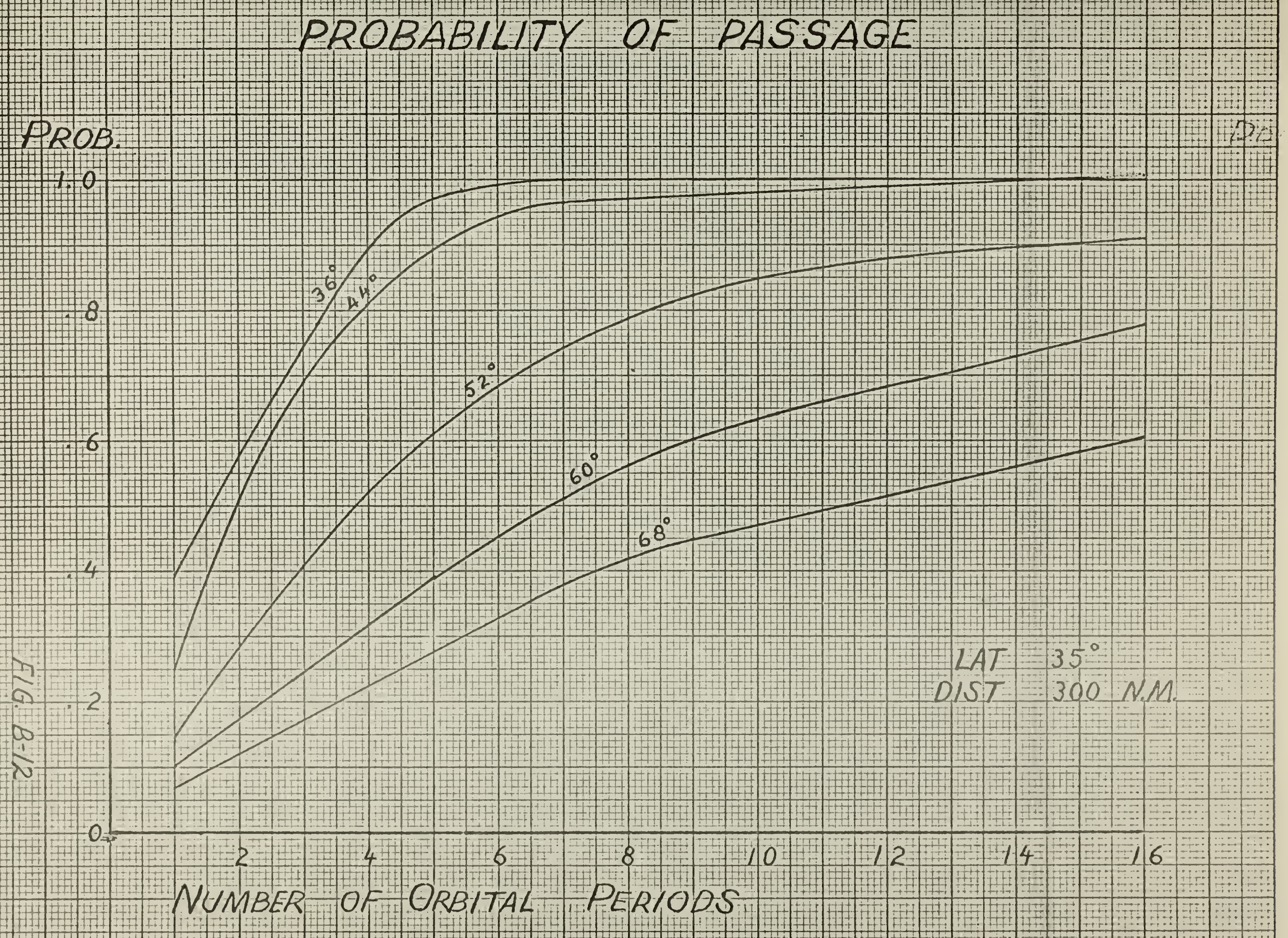


FIG. B-11



PROBABILITY OF PASSAGE
PROB.
1.0
.8
.6
.4
.2
0
2
4
6
8
10
12
14
16
NUMBER OF ORBITAL PERIODS
36°
44°
52°
60°
68°
LAT 35°
DIST 300 N.M.

FIG. B-12

PROBABILITY OF PASSAGE

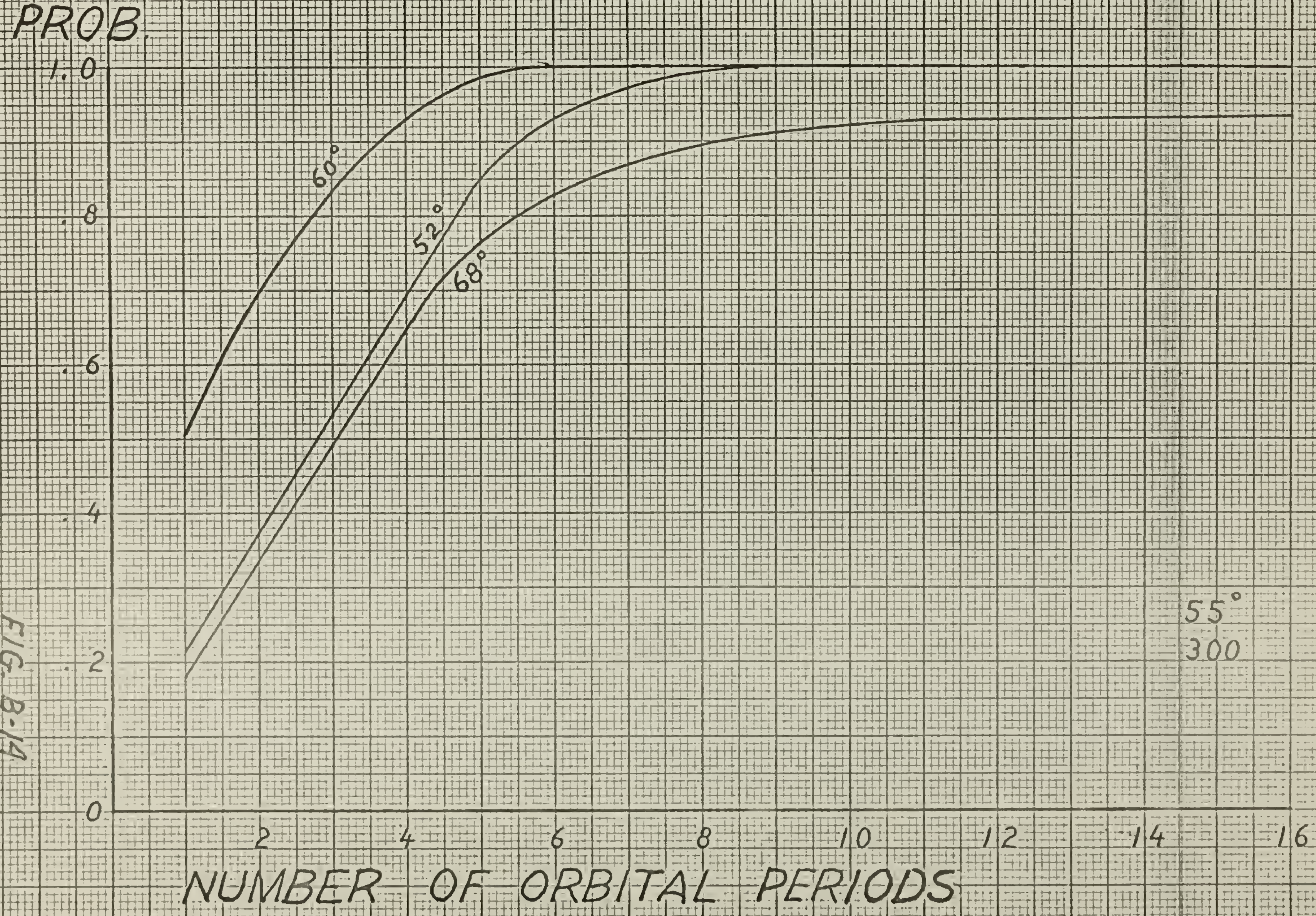


FIG. B-14

PROBABILITY OF PASSAGE

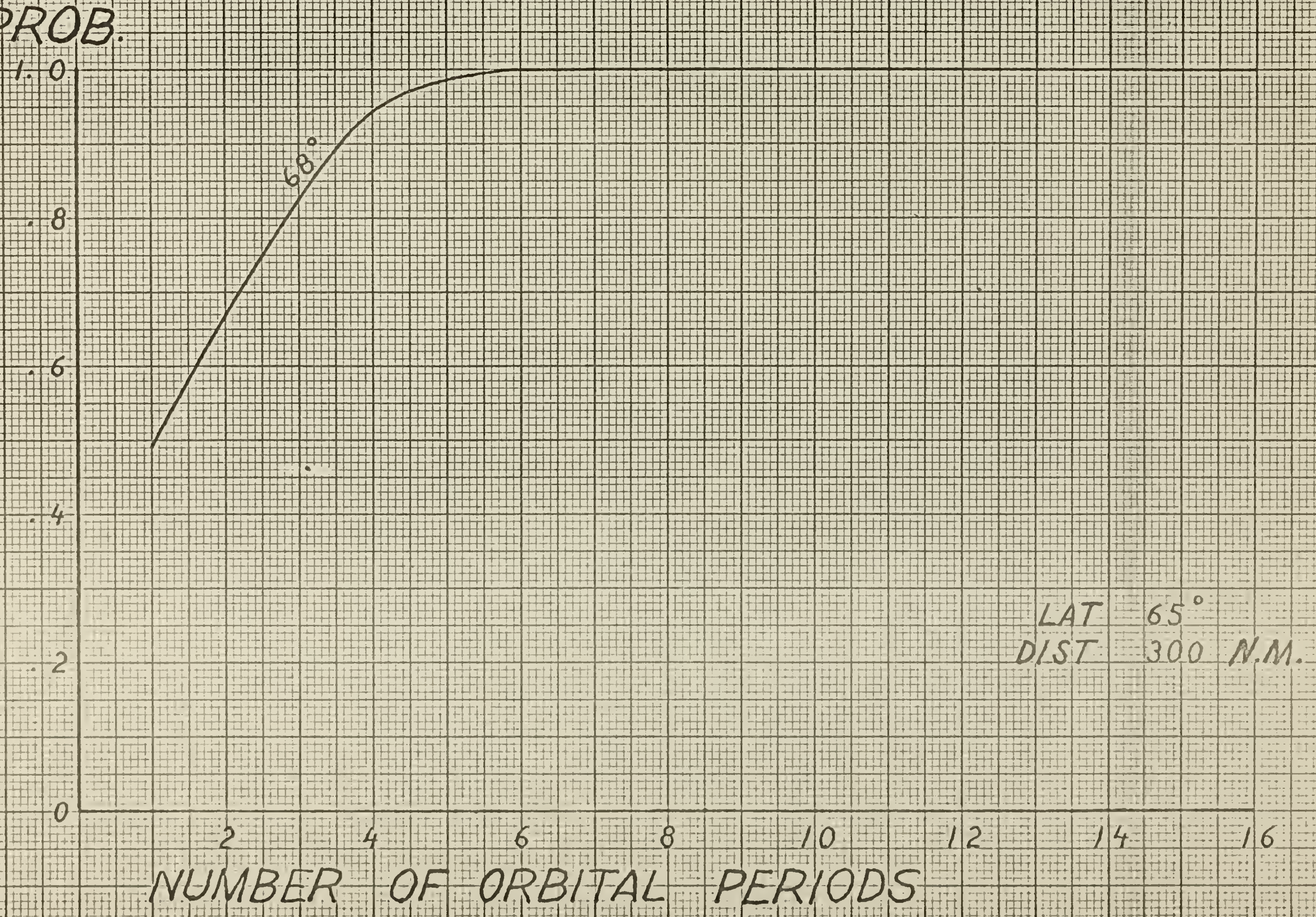


FIG. B-15